CONCERNING LIMITING SETS IN ABSTRACT SPACES*

BY

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By a *limit point* of a point set M we mean a point which is the *limit*† of an infinite sequence of distinct points belonging to M. This notion may be generalized by regarding a *point set* as being the limit, in some sense, of a *collection* of point sets. We shall give a precise formulation of such a generalization: A point set K is said to be *the limiting set* of a collection G of point sets provided that it is the set of all points P such that there exists a sequence of points $P_1, P_2, P_3, \cdots$, and a sequence $g_1, g_2, g_3, \cdots$, of *distinct* elements of G such that P is the limit of the sequence $P_1, P_2, P_3, \cdots$, and for every n, P_n belongs to g_n.‡ Theorems concerning such limiting sets are frequently

applicable in point set theory and in abstract topology; as an example we may mention theorems concerning conditions under which the limiting set is connected (cf. Theorems 11 and 12, and the discussion and references there given).

It is of interest to discover what properties hold for limiting sets in euclidean spaces, and to determine the general classes of abstract spaces in which such theorems hold. One of the purposes of the study of abstract spaces is the discovery of conditions, preferably necessary and sufficient, that a given theorem should hold in that space. As an example we may consider the result shown in Theorem 1 of this paper. In ordinary space the limiting set of every collection of point sets is closed. We show in Theorem 1 that in a space L a necessary and sufficient condition that the limiting set of every infinite collection of point sets be closed is that the space under consideration be a space S.* Thus, the majority of the properties of a euclidean space have little bearing on the fact that in such a space the limiting set of a collection of point sets is closed.

Another example of such a study is the one we make of the distributive property. A space is said to have *the distributive property*† or *to be distributive* provided that if in that space K is a closed point set and G is a collection of point sets and each point of K belongs to some sub-set of K which is the limiting set of some sub-collection of G, then K is itself the limiting set of a sub-collection G_1 of G. If it be specified in addition that G_1 be countable, the space is said to have the *countably distributive property* or to be *countably distributive*. We show in this paper that every distributive space D is locally compact‡ and further that in order that a space D have the countably

zewski, *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, (2), vol. 16 (1912), pp. 93–94; F. Hausdorff, (I), *Grundzüge der Mengenlehre*, Leipzig, Veit, 1914, pp. 233–239 and 296–304, and (II), *Mengenlehre*, Berlin, de Gruyter, 1927, pp. 145–150; L. Vietoris, *Stetige Mengen*, Monatshefte für Mathematik und Physik, vol. 31 (1921), pp. 184–194; Zoretti-Rosenthal, *Mengen die von einem Parameter abhangen*, Encyklopaedie der Mathematischen Wissenschaften, Band II, 3, Heft 7 (II C 9a), §15, pp. 938–941.

* The *derived set* of a point set is the set of all its limit points. A *space S* is a space L in which the derived set of every point set is closed. Cf. M. Fréchet, (IV), *Relations entre les notions de limite et de distance*, these Transactions, vol. 19 (1918), p. 55. If M is a point set, M' denotes the derived set of M, and $\overline{M} = M + M'$. $M \subset N$ means that M *is a sub-set of N*.

† The term is due to R. L. Moore. The question whether a bounded euclidean space has this property was raised by W. L. Ayres during a discussion of the proof of a special case of my Theorem 13, and was settled in the affirmative by R. L. Moore.

‡ A point set is said to be *compact* provided that each of its infinite sub-sets has at least one limit point. (Cf. Fréchet I, pp. 6–7.) A space D is said to be *locally compact at a point P* or to be *compact near P* provided that P belongs to a compact open set. A point set is said to be an *open set* provided that its complement in the space under consideration is closed. A space is *locally compact* provided that it is locally compact at each of its points. A point set is *separable* provided that it

distributive property it is necessary and sufficient that it be locally compact and separable.

Certain theorems concerning limiting sets have an analogy to covering theorems. The question concerning the possibility of replacing any collection of point sets in a space by a countable sub-collection having the same limiting set is analogous to the one whether the Lindelöf property* holds in the space. This question is, however, of less interest than the following one: Under what conditions is a point set the limiting set of a given collection of point sets? (See Theorems 9, 10, 12, 13, and 14.) The distributive property is of especial interest in this connection; in a space possessing this property we have a condition which may be described roughly as follows: If the property of being the limiting set of a sub-collection of G is distributed everywhere in a closed point set K, then K itself has this property. It was for the reason just indicated that the term "distributive property" suggested itself to Moore.

Return to the consideration of a hyperspace whose elements are point sets in a given space. Condition (2) mentioned in the definition of a space L is not satisfied if we interpret "the limit of a sequence" as the limiting set of a sequence of point sets.† This objection does not apply to the sequential limiting set. The limiting set of an infinite collection of point sets is called *the sequential*‡ *limiting set* of this collection if and only if every infinite sub-collection of the given collection has the same limiting set; a collection of point sets is *convergent* if it has a non-vacuous sequential limiting set and is *divergent* if it has a vacuous sequential limiting set.§ In Theorems 16 and 17 we show that the aggregate of all closed point sets in an abstract space may be regarded as a space L provided that we introduce a suitable definition for "the limit of a sequence," which involves the notion of a sequential limiting set. In this connection Theorems 5, 6, and 7 are of interest; Theorem 6 shows that there exist spaces L and even spaces D in which no collection of point sets contains a convergent or a divergent sub-sequence.

Janiszewski‖ gives a definition of the limit set of a sequence of point sets;

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contains a countable sub-set such that every point of the set either belongs to this sub-set or is a limit point of it. Cf. Fréchet III, p. 341.

* A collection of open sets is said to *cover* a point set K provided that each point of K belongs to some element of the given collection. A space is said to have the *Lindelöf property* provided that for every collection of open sets covering a point set in it there exists a countable sub-collection covering the point set. Lindelöf first showed that a euclidean space has this property; cf. his paper, *Remarques sur un théorème fondamental de la théorie des ensembles*, Acta Mathematica, vol. 29 (1905), p. 188.

† Cf. Janiszewski, loc. cit., pp. 94–95, example 1.

‡ Cf. R. L. Moore, loc. cit., and Hausdorff I, loc. cit.

§ Cf. Hausdorff I, pp. 230 and 232.

‖ Loc. cit., p. 94.

we modify his definition as follows: The *limit set* of a collection of point sets is the set of all points Q such that Q belongs to the limiting set of every infinite sub-collection of the given collection. It follows from our definition that if P is a point in the limiting set of a collection G of point sets, then P belongs to the *limit* set of some infinite sub-collection of G; we show in Theorem 6 that it is not true in general, even in a space D, that P belongs to the *sequential limiting set* of a sub-collection of G.

Janiszewski* shows that the limit set may be a proper sub-set of the limiting set, and may even be vacuous. In fact, the sequential limiting set exists if and only if the limit set is the same as the limiting set; and, in case this is true, the three are identical. The limiting set of an infinite sub-collection of a collection of point sets is a sub-set of the limiting set of the collection. It is not implied in our definitions that all the elements of a collection of point sets should be distinct; it is possible, for instance that each element should consist of a single point, and that all are the same point; in this case the limiting set of the collection would be the given point.

In concluding the introduction the author wishes to thank R. L. Moore, who indicated to one of his classes the desirability of a systematic study of limiting sets, and whose criticisms and suggestions have been of great value in the preparation of this paper.

THEOREM 1. *In order that every collection of point sets in a space L should have a closed limiting set it is necessary and sufficient that the space be a space S.*

If the condition is not sufficient, there must exist a space S in which there exists a collection G of point sets whose limiting set is not closed. Let P be a point which does not belong to K, the limiting set of G, but is the sequential limit point† of a sequence $U = Q_1 + Q_2 + Q_3 + \cdots$ of distinct points belonging to K. Let F_n be the aggregate of all elements g of G such that, for some integer $m > n$, g contains the point Q_m. It follows that $F_{n+1} \subset F_n$. Then for some sufficiently large value N of n, F_N will contain at most a finite number of elements; otherwise, as may easily be seen, P would have to belong to the limiting set of G. Let $F = G - F_N$, $P_i = Q_{N+i}$, and $T = \sum P_i (i = 1, 2, 3, \cdots)$.

It follows from the definition of P_i and the fact that P_i is a point of no element of F that there exists for each positive integer i a sequence

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* Loc. cit.

† A point is the *sequential limit point* of a sequence of points M provided that it is the limit of every infinite sub-sequence of M. Cf. R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 134.

$T_i = \sum P_{ji}(j = 1, 2, 3, \cdots)$ of distinct points such that (1) P_i is the sequential limit point of T_i and (2) for each integer i, P_{ji} belongs to an element g_{ji} of F such that if $m \neq n$ then $g_{mi} \neq g_{ni}$.

Let $S_1 = Q_{11} = P_{11}$ and let $h_{11} = G_1$ be a definite element of F containing P_{11}. For $k > 1$ let $G_k = \sum h_{ki}(i = 1, 2, 3, \cdots, k)$ be a sequence of elements of F and $S_k = \sum Q_{ki}(i = 1, 2, 3, \cdots, k)$ be a sequence of points selected as follows: h_{ki} is an element of $\sum g_{ji}(j = 1, 2, 3, \cdots, \infty)$ with $j = n_{ki}$, where n_{ki} is the smallest positive integer x such that g_{xi} is not one of the finite number of elements of F belonging to $G_1, G_2, \cdots, G_{k-1}$; Q_{ki} is the point $P_{n(k,i),i}$.* Let $W = \sum S_i(i = 1, 2, 3, \cdots, \infty)$. For every integer i, W contains infinitely many points in common with T_i, and hence T is a sub-set of the derived set of W. Since W is a sub-set of a space S, its derived set is closed. It follows that there exists a set $Q_{k(1),i(1)}, Q_{k(2),i(2)}, Q_{k(3),i(3)}, \cdots$, of distinct points of W having P as a sequential limit point. But no element of G belongs to more than one of the collections G_i, each of which contains only a finite number of elements. Hence the sequence $h_{k(1),i(1)}, h_{k(2),i(2)}, h_{k(3),i(3)}, \cdots$ contains infinitely many distinct elements of G, and P belongs to the limiting set of this sequence and thus to the limiting set of G. Thus we have demonstrated the sufficiency of the condition.

Suppose that there exists a space L in which the limiting set of every collection of point sets is closed, but which is not a space S. Then there exists in this space a point set M whose derived set has a limit point P which does not belong to M'. Let $W = P_1 + P_2 + P_3 + \cdots$ be a sequence of distinct points belonging to M' and having P as a sequential limit point. Then for each positive integer i there exists a sequence $W_i = P_{1i} + P_{2i} + P_{3i} + \cdots$ which has P_i as a sequential limit point, is a sub-set of M, and does not contain P. Let G be a collection whose elements are P_{xy}, where x and y are independent variables, each having as range the set of positive integers. Then W is a sub-set of the limiting set of G, and so is P, since the limiting set is closed. It follows that P is a limit point of M, contrary to its definition. Thus the condition is necessary.

DEFINITION. A point P is said to be a *point of condensation*† of a point set M provided that every open set containing P contains uncountably many points of M.

THEOREM 2. *In order that it should be true for each infinite collection of point sets in a space D that at most a countable number of its elements fail to*

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* For typographical reasons when n_i and n_{ij} occur as subscripts or superscripts they will be printed $n(i)$ and $n(i, j)$ respectively.

† Cf. Lindelöf, loc. cit., p. 184, and Fréchet I, p. 6.

be sub-sets of its limiting set, it is necessary and sufficient that the space be separable.

We shall show first that the condition is sufficient. Let K be the limiting set of a collection G of point sets in a separable space D. Suppose that there exists an uncountable sub-collection $\overline{G}$ of the elements of G such that no element of $\overline{G}$ is a sub-set of K. There must exist a positive number y such that there exist uncountably many elements of $\overline{G}$ containing points whose distances* from K are greater than y. Let E be the set of all points belonging to elements of $\overline{G}$ and having distances from K greater than y. If a point of E is common to infinitely many elements of $\overline{G}$ it belongs to the limiting set of G. This is clearly impossible. Further, if E contains at most a countable number of points, it follows from what we have just proved that at most a countable number of elements of $\overline{G}$ contain points in E; this involves a contradiction with our definition of E. Hence E contains uncountably many points, and has a point of condensation.† Since no point of E belongs to more than a finite number of elements of $\overline{G}$, this point of condensation belongs to the limiting set of G. Since its distance from K is not less than y we have a contradiction. Hence $\overline{G}$ does not exist, and the condition given in the statement of the theorem is sufficient.

We shall show next that the condition is necessary. Suppose that it is not, and that there exists a non-separable space D in which every infinite collection of point sets has the property mentioned in the statement of the theorem. Then the space contains an uncountable point set H having no limit point.‡ Let G be a collection of elements g such that each g consists of a single point belonging to H and each point of H belongs to a single element of G. It follows that H' is the limiting set of G. From the fact that all except a countable number of the elements of G are sub-sets of H' it follows that H' is non-vacuous. Thus, the supposition that the space is not separable has led to a contradiction.

DEFINITION. A space is said to have the *Cantor-Bendixson property* provided that every closed point set in it is the sum of a perfect set and a countable set. The fact that this property holds in a euclidean space is, as

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* If P is a point and N is a point set, by $d(P, N) = d(N, P)$, *the distance from P to N, is meant* the lower bound of the set of values $[d(P, Q)]$, where Q is a point of N.

† Cf. W. Gross, *Zur Theorie der Mengen in denen ein Distanzbegriff definiert ist*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, vol. 123 (1914), IIa, pp. 805–806, and Fréchet I, p. 6.

‡ Cf. Gross, loc. cit., pp. 805–806.

indicated by Lindelöf, loc. cit., the most important part of a theorem by Cantor* and Bendixson.

THEOREM 3. *In order that a space D should have the Cantor-Bendixson property it is necessary and sufficient that the space be separable.*

The sufficiency of the condition follows from the work of Fréchet† and Gross†; the argument given by Fréchet is similar to one given for the euclidean case by Lindelöf (loc. cit., pp. 183, 187). The necessity of the condition follows easily from Gross's work. Suppose that there exists a space D which has the Cantor-Bendixson property but is not separable. Then it contains an uncountable point set M which has no limit point;‡ then M is closed. It follows that M is the sum of a perfect set and a countable set. Since M contains uncountably many points, this perfect set is non-vacuous. Every point of the perfect set is a limit point of this set and of M. Thus we get a contradiction.

THEOREM 4. *In order that every infinite collection of point sets in a space D should contain a countable sub-collection having the same limiting set as the collection itself it is necessary and sufficient that the space be separable.*

It was pointed out by W. L. Ayres that the sufficiency of the condition for a bounded euclidean space is established as a consequence of the fact that such a space has the countably distributive property. The same conclusion holds for any space D that has this property. Not all separable spaces D have this property, however; cf. Theorem 9.

Let G be a collection of point sets in a separable space D, and K be its limiting set. Since the space is separable there exists a countable§ sequence $P_1+P_2+P_3+ \cdots$ every element of which is a point of K, and such that every point of K is the sequential limit point of a subsequence of it. For each positive integer n there exists an element g_n of G whose distance from P_n is less than $1/n$ and which is distinct from $g_1, g_2, g_3, \cdots, g_{n-1}$, the elements associated respectively with $P_1, P_2, P_3, \cdots, P_{n-1}$. Then K is the limiting set of $g_1+g_2+g_3+ \cdots$.

The necessity of the condition may be established as follows: Let G be a collection of point sets in a space D such that (1) each element of G is a

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* Cf. G. Cantor, *Fondements d'une théorie générale des ensembles*, Acta Mathematica, vol. 2 (1883), pp. 405, 409–414; and I. Bendixson, *Quelques théorèmes de la théorie des ensembles de points*, Acta Mathematica, vol. 2 (1883), pp. 415–427.

† Cf. Fréchet I, p. 19, and Gross, loc. cit., pp. 805–806, 815–816.

‡ Cf. Gross, loc. cit., pp. 805–806.

§ Cf. Gross, loc. cit., pp. 805–806, and Fréchet I, p. 27.

single point of the space and (2) for each point x of the space there exists a countable sequence g_{1x}, g_{2x}, g_{3x}, $\cdots$ of elements of G, each of which is the point x. Clearly the limiting set of G is M, the set of all points in the space. By our hypothesis G contains a countable sub-collection G_1 which has the same limiting set. It follows from our definition of a limiting set that if y is a point of M and is not the limit of an infinite sequence of distinct points belonging to M_1, the set of all points belonging to elements of G_1, it is the limit of a sequence y, y, y, $\cdots$ of points of M_1, each of which is the point y. Then y belongs to M_1. Since M_1 is countable, M is separable.

THEOREM 5. *If, in a space D, the limiting set of a collection G of point sets is separable, then every infinite sub-collection of G contains a sub-collection that has a sequential limiting set.*

The question as to the truth of this theorem for a bounded euclidean space was raised by R. L. Moore, and in June, 1925, was proved independently by him and by me; he pointed out that my method of proof has an advantage over his in that it applies to abstract spaces. After submitting this paper for publication I learned that T. Ważewski has the following theorem in his thesis:[*] In a compact space D every infinite collection of point sets contains a convergent sub-collection. Also, since this paper was submitted for publication Zarankiewicz[†] and Hausdorff have published analogous results, Zarankiewicz showing that in a euclidean space and Hausdorff that in a separable space D every infinite collection of point sets contains a sub-collection which has a sequential limiting set. The argument given by Hausdorff does not suffice to prove the theorem we have stated, since he uses the fact that the complement of the limiting set is separable; however, if the proof given by Zarankiewicz, which is similar to the one discovered by the author, is modified slightly, it suffices for our purposes.

In 1909 Zoretti[‡] proved the following theorem: If, in a plane, G is a collection of continua[§] (1) whose sum is bounded, (2) whose limit set is non-vacuous, and (3) whose limiting set is a continuum, then G contains an in-

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[*] *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan*, Krakow, 1923, p. 24.

[†] Cf. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 127–129; and Hausdorff II, pp. 147–148.

[‡] Cf. Zoretti II.

[§] A *continuum* is a closed and connected point set. (Cf. Cantor, loc. cit., p. 406.) A point set is *connected* provided that it is not the sum of two mutually exclusive, non-vacuous point sets, neither of which contains a limit point of the other. Cf. N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), p. 303, and Hausdorff I, p. 244.

finite sub-collection which has a closed and connected sequential limiting set. The conditions (2) and (3) are unnecessary, for the existence of a sequence having property (2) follows from (1), and (3) follows from (2) by a theorem of Zoretti's, p. 116, loc. cit. In his process of selecting a sequence which has a sequential limiting set, Zoretti makes no use of the fact* that the elements of his collection are continua, and this process is adequate to prove that every infinite collection of point sets in a plane whose sum is bounded contains a convergent sub-collection.

Proof of Theorem 5. Let K be a separable point set in a space D and G be a collection of point sets having K as a limiting set. Then K has a countable sub-set H such that if X is a point of K it is either a point of H or a limit point of H. For each point of H consider all the spheres† having this point as a center and having radii whose lengths are rational numbers. Let $R_1, R_2, R_3, \cdots$ be the set of the interiors of these spheres. Now proceed as in the paper of Zarankiewicz (loc. cit., pp. 128–129).

THEOREM 6. *In a space L which contains a point set having the power of the continuum and having a vacuous derived set there exists an infinite collection of point sets which has no convergent or divergent sub-collection.*

The following example shows that there exist spaces D which satisfy the hypothesis of the theorem: Let D_1 be an aggregate of objects having the power of the continuum in which all the requirements of a space D are satisfied, subject to the condition that the distance between two distinct elements of D_1 is unity.

Proof of Theorem 6. Consider a space L in which there exists a point set M having the power of the continuum, but having no limit point. Let X be the set of all real numbers x such that $0 < x < 1$. Then there exists a correspondence such that to each x there corresponds exactly one point P_x of M and conversely. Let the number x be expanded as an *infinite* decimal in the binary scale. Let G be a collection of point sets, $g_1, g_2, g_3, \cdots$, where the set g_n consists of all points of M for which the corresponding elements in X, when given the binary expansion just mentioned, have as their nth digit unity.

Let $T = g_{n(1)} + g_{n(2)} + g_{n(3)} + \cdots$ be a sub-collection of G such that $n_1 < n_2 < n_3 < \cdots$. Let $z = \sum 2^{-n(2i)}$ ($i = 1, 2, 3, \cdots$), and $H = \sum g_{n(2i)}$ ($i = 1, 2, 3, \cdots$). Then P_z is common to all the elements of H and hence belongs to the limiting set of H. Hence the limiting set of T is non-vacuous, and T

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* This fact is used in proving that the limiting set is connected.

† The terms *sphere, interior* and *exterior of a sphere*, etc. are defined precisely as in euclidean geometry. Cf. Fréchet I, p. 21, paragraph 34. A *spherical region* is the interior of a sphere.

is not divergent. It follows from our definitions of M and G that a point of M belongs to the limiting set of an infinite sub-collection of G if and only if the point belongs to infinitely many elements of this sub-collection. Hence P_z does not belong to the limiting set of $T - H$. Hence T does not converge.

THEOREM 7. *If the hypothesis of the continuum* is correct, then in order that every infinite collection of point sets in a space D should contain a convergent or a divergent sub-collection, it is necessary and sufficient that the space be separable.*

The theorem is a consequence of Gross, pp. 805-806, Fréchet I, p. 27, and Theorem 6, and Hausdorff II, pp. 147–148.

THEOREM 8. *Every space D which has the distributive property is locally compact.*

If the theorem is not true, there exists a space D having the distributive property, and containing a point P near which it is not compact and a countably infinite set of points, K, which has no limit point and does not contain P. Let R_1 be a spherical region containing P such that $\overline{R}_1$ contains no point of K. There exists in R_1 a countable point set M_1 having no limit point and not containing P; within R_1 there is a spherical region R_2 with its center at P, of diameter less than one half, and containing no point of M_1. This process may be continued; in general, let R_n be a spherical region of diameter less than $1/n$, which contains no point of the point sets $M_1, M_2, M_3, \cdots$, M_{n-1}, has its center at P, and lies within the region R_{n-1}; let $M_n = P_{1n} + P_{2n} + P_{3n} + \cdots$ be a countable sequence of points which lies within R_n, does not contain P, and has no limit point. Let $K = P_1 + P_2 + P_3 + \cdots$. For every pair of integers i and j let $g_{ij} = P_{ij} + P_j$. Then P_j is the limiting set of $\sum g_{ij}$ ($i = 1, 2, 3, \cdots, \infty$). From the fact that the space under consideration has the distributive property and that K is closed, since it has no limit point, it follows that K is the limiting set of a sub-collection N of $\sum \sum g_{ij}$ (i and $j = 1, 2, 3, 4, \cdots, \infty$). Then there exists an integer m such that R_m contains no points belonging to elements of N. It follows that N is a sub-collection of $\sum \sum g_{ij}$ ($j = 1, 2, 3, \cdots, m-1; i = 1, 2, 3, \cdots, \infty$). There exists a spherical region R which contains P_{m+1} but contains no point of $K - P_{m+1}$ or of $\overline{R}_1$. But P_{m+1} does not belong to any point sets in the collection $\sum \sum g_{ij}$ ($j = 1, 2, 3, \cdots, m-1; i = 1, 2, 3, \cdots, \infty$). It follows

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that P_{m+1} does not belong to the limiting set of N. Thus, the supposition that the space is not locally compact has led to a contradiction.

THEOREM 9. *In order that a space D should have the countably distributive property it is necessary and sufficient that it be locally compact and separable.*

First we shall show the sufficiency of the condition. Let K be a closed point set in a separable, locally compact space D and G be a collection of point sets such that each point of K belongs to a sub-set of K which is the limiting set of some infinite sub-collection of G. Let M be the limiting set of G and $N = M - K$. The point set K is separable* and hence there exists a countable sub-set $H = P_1 + P_2 + P_3 + \cdots$ of K such that $K \subset \overline{H}$. For each integer i there exists a sub-collection $G_i = g_{1i} + g_{2i} + g_{3i} + \cdots$ of G such that the *limit set* of G_i is a sub-set of K and contains P_i.

From the fact that K is closed and that the space is locally compact it follows that for each point Q of N there exists a spherical region R_Q which is compact and such that $\overline{R}_Q$ contains no point of K. From the fact that the space is separable and hence has the Lindelöf property† it follows that there exists a countable sequence $R_1, R_2, R_3, \cdots$ of these regions covering N. If i and j are integers, the region R_j contains points of at most a finite number of elements of G_i; for otherwise, since $\overline{R}_j$ is compact, it would contain a point of the limiting set of G_i, thus contradicting the fact that the limiting set of G_i is a sub-set of K and $\overline{R}_j$ contains no point of K. Hence, for every integer i the sequence G_i contains an infinite sub-sequence F_i such that no element of F_i contains a point in any of the regions $R_1, R_2, R_3, \cdots, R_i$. Let $F = \sum F_i$ $(i = 1, 2, 3, \cdots, \infty)$. We shall show that E, the limiting set of F, is a sub-set of K. If it is not, it must contain a point W of N, since F is a sub-collection of G and E is then a sub-set of M. There exists an integer m such that W belongs to R_m. Let $n \geq m$. Then it follows from the definition of F_n that F_n contains no point within R_m. Further, as we have pointed out above, if $1 \leq p \leq m$, at most a finite number of the elements of G_p, and hence of F_p, contain points within R_m. Hence R_m contains points of at most a finite number of elements of F, and hence W cannot belong to the limiting set of F.

We shall show next that K is a sub-set of the limiting set of F. It follows from the definition of F that H is a sub-set of the limiting set of F. If X is a point of $K - H$, it is a limit point of H. By Theorem 1, X is then a point of the limiting set of F. From $E \subset K$ and $K \supset E$ it follows that $K = E$.

Given a space D which has the countably distributive property. If G

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* Cf. Fréchet I, p. 27, and Gross, pp. 805–806.

† Cf. Gross, pp. 805, 806, 809.

is a collection of point sets, it follows by Theorem 1 that its limiting set is closed. By our hypothesis G can be replaced by a countable sub-collection having the same limiting set. By Theorem 4 the space is separable. By Theorem 8 the space is locally compact. Hence, the condition is necessary.

DEFINITIONS. If M and N are two point sets in a space D, by $l(M, N)$, *the lower distance from* M *to* N, is meant the lower bound of the values $d(P, N)$, where P is a point of M; and by $u(M, N)$, *the upper distance from* M *to* N, is meant the upper bound of the values $d(P, N)$. If G and H are collections of point sets, we shall say that H is *upper semi-continuous with respect to* G provided that if g is a fixed element of G and h is a variable element of H, then as $l(h, g)$ approaches zero so does $u(h, g)$. If the collection G is upper semi-continuous with respect to itself, it is said to be an *upper semi-continuous* collection*.

THEOREM 10. *If in a space D, K is a closed point set and G is a collection of point sets, then a sufficient condition that K be the limiting set of G is that K is a sub-set of the limiting set of G and (1) contains points in common with the limiting set of every infinite sub-collection of G that has a non-vacuous limiting set; (2) G is upper semi-continuous with respect to the point set K. If it be specified in addition that the space be compact, the condition mentioned above is both necessary and sufficient.*

It is easy to construct examples showing that the condition mentioned in the theorem does not remain necessary in case the stipulation that the space be compact be removed. The theorem remains true if we replace (1) by (1') following: (1') *If e is a positive number, there exist at most a finite number of elements g of G such that $l(g, K) > e$.*

Proof of Theorem 10. Let H be the limiting set of G. If H contains a point x not belonging to K, there must exist an infinite sub-sequence $G_x = g_1 + g_2 + g_3 + \cdots$ of G such that every element of this sequence is at a distance from x less than $\frac{1}{2}d(x, K) > 0$, and for each positive integer i, $d(x, g_i) < 1/i$. By the hypothesis there exist (1) a positive number d such that when $l(g, K) < d$ then $u(g, K) < \frac{1}{2}d(x, K)$, and (2) an infinite sub-sequence of elements of G_x, every element of this sub-sequence being at a lower distance from K less than d. Thus the supposition that the point x exists has led to a contradiction, and $K = H$.

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* For definitions of lower distance, of upper distance, and of upper semi-continuous collections of point sets, cf. R. L. Moore, *Concerning upper semi-continuous collections of continua which do not separate a given continuum*, Proceedings of the National Academy of Sciences, vol. 10 (1924), p. 356, and *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), p. 416.

We shall now show that if the space is compact the condition is necessary. Part (1) of the condition follows immediately. If part (2) does not follow, there must exist a positive number ϵ and a sub-sequence $H = H_1 + H_2 + H_3 + \cdots$ of elements of G such that, for every n, $u(H_n, K) > \epsilon$, but $l(H_n, K)$ approaches zero as n approaches infinity. Since the space is compact it follows that the limiting set of H contains a point x whose distance from K is not less than ϵ. But the result just stated involves a contradiction of our assumption that K is the limiting set of G and hence contains the limiting set of H.

DEFINITIONS. If x and y are a pair of points, a sequence of points $x_1 (=x)$, $x_2, x_3, \cdots, x_n (=y)$ is said to be *a chain joining x and y*; a pair of consecutive points, x_j and x_{j+1}, is said to be a *link* of this chain and $d(x_j, x_{j+1})$ is the *length* of this link. If g is a point set let $f(g)$ be the lower bound of the set of all positive numbers h such that any pair of points in g can be joined by a chain every point of which belongs to g and every link of which has a length less than or equal to h. If g is a bounded point set, $f(g)$ certainly exists; otherwise it need not. If G is *zusammenhängend*[*] in the sense of Cantor, $f(g) = 0$.

Let M be a point set in a space L and P and Q be points of M. Let $C_{PQ} = \sum \sum P_{ij}$ $(j = 1, 2, 3, \cdots, \infty; i = 1, 2, 3, \cdots, n_j)$ be a sub-set of M such that for each positive integral value of j, $P_{1j} = P$, n_j is a positive integer, and $P_{n(j),j} = Q$. The points P and Q are said *to be well chained in M* provided that there exists at least one C_{PQ} such that if $j_1 < j_2 < j_3 < \cdots$ and the sequence $\sum P_{m(j(k)),j(k)}$ $(k = 1, 2, 3, \cdots, \infty)$, of the set C_{PQ} has a sequential limit point T, then T is a sequential limit point also of every infinite sub-sequence of $\sum P_{m(j(k))+1,j(k)}$ $(k = 1, 2, 3, \cdots, \infty)$ and of $\sum P_{m(j(k))-1,j(k)}$ $(k = 1, 2, 3, \cdots, \infty)$, where $1 \leqq m_{j(k)} \leqq n_{j(k)}$. If every pair P and Q of points of M is well chained in M, M is said to be *well chained*.

THEOREM 11. *If, in a locally compact space D, $G = [g]$ is a collection of point sets, K is the limiting set of G, and $f(g)$ approaches zero as $l(g, K)$ approaches zero, then in order that K be well chained it is necessary and sufficient that there exist a well chained sub-set K_1 of K which contains points in common with every sub-set of K which is the limiting set of a sub-collection of G.*

Janiszewski considers an analogous question (1) for a bounded number space and (2) in an abstract space which may be characterized as a perfectly compact, connected space in which a distance is defined (cf. Janiszewski, p. 162, loc. cit.); he shows that in such spaces a sufficient condition that

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[*] Cf. Cantor, loc. cit., p. 406, and *Ueber unendliche, lineare Punktmannigfaltigkeiten*, Mathematische Annalen, vol. 21 (1883), p. 576.

the limiting set be *connected* is that the *limit set exist.* Our theorem is more general than his in that we give necessary and sufficient conditions and that our space is more general.

Schoenflies (loc. cit.) gives a less general case than Janiszewski; he assumes that the sequential limiting set exists. See also Zoretti I, pp. 8–9, and Zoretti II. In connection with Zoretti's work see our comments following Theorem 5. See also Vietoris, loc. cit., p. 186, and Hausdorff I, pp. 301–302.

It might be pointed out that in a non-compact space D the limiting set of a collection of connected point sets need not be connected, cf. Hausdorff I, p. 239, last four lines. As an example of a bounded space D where the condition given in Theorem 11 is not sufficient consider the space D_3 given below. Note, however, that this space is not locally compact. Let H_x be the straight line interval in the euclidean plane having as end points $(x, 0)$ and $(x, 1)$. Let A be the point $(0, 0)$ and B the point $(0, 1)$. Let the points of D_3 be the points in the euclidean sense of $A + B + \sum H_{1/i}$ ($i = 1, 2, 3, \cdots, \infty$), and distance and limit be defined as in ordinary space. Let G be a collection of point sets whose elements are the H's in $\sum H_{1/i}$ ($i = 1, 2, 3, \cdots, \infty$). Then G is a collection of connected point sets having as a sequential limiting set the sum of A and B. The set $A + B$ is neither connected nor well chained.

Proof of Theorem 11. Obviously the condition is necessary; we shall prove its sufficiency.

First we shall show that if M is a connected point set it is "zusammenhängend." For, if it is not, there exists a positive number d and a pair of points x and y of M which cannot be joined by a finite chain of points belonging to M, each link in the chain having a length less than d. Let N be the set of all points of M which can be joined to x by a chain such as that just mentioned. Then both N and $M - N$ are non-vacuous and, since M is connected, there exists a point z which belongs to one of the point sets N and $M - N$ and is a limit point of the other. Let E be a spherical region with center at z and having a radius of length $d/3$. It follows that E contains points of both N and $M - N$. Clearly this involves a contradiction. Evidently M is well chained.

We shall show next that if M and N are two non-compact, connected point sets their sum is well chained. Let $m_1, m_2, m_3, \cdots$ and $n_1, n_2, n_3, \cdots$ be two infinite sequences of points belonging to M and N respectively, and neither of which has a limit point. It follows from the preceding paragraph that for each positive integer i there exists a pair of chains of points $m_{1i}(=m_1)$, $m_{2i}, m_{3i}, \cdots, m_{p(i),i}(=m_i)$ and $n_{1i}(=n_1), n_{2i}, n_{3i}, \cdots, n_{q(i),i}$ ($=n_i$) be-

longing to $\dot{M}$ and to N respectively such that each link in each of these chains has a length less than $1/i$. Let C_i be the chain m_{1i}, m_{2i}, m_{3i}, $\cdots$, $m_{p(i),i}$, $n_{q(i),i}$, $\cdots$, n_{2i}, n_{1i}. It follows from the definitions of C_1, C_2, C_3, $\cdots$ that m_1 and n_1 are well chained in M, and since they may be selected at pleasure it follows that $M+N$ is well chained.

Hence, to prove that K is well chained it suffices to prove that either (1) K is connected or (2) every maximal* connected sub-set of K is non-compact. Suppose that neither of these conditions is realized. Then K has a compact maximal connected sub-set T which does not contain all of K. About each point of T there exists a spherical region such that the sum of its points and limit points is compact. The point set T can be covered by a finite† sub-collection of this collection of regions. Let H be the sum of the points belonging to this finite set of regions and B be the boundary‡ of H. Evidently B contains no point of T. Since $\overline{H}$ is the sum of a finite number of compact point sets, it is compact. Let G_2 be a sub-collection of G having a *limit* set containing a point P of T. Suppose that K_2, the *limiting* set of G_2, contains points without $\overline{H}$; let Q be such a point. Let d be a positive number which is less than each of the numbers $d(P, h)$ and $d(Q, H)$.§ It follows from the hypothesis of the theorem and the definition of G_2, that G_2 contains a sub-sequence g_1, g_2, g_3, $\cdots$ such that for each positive integer n, $f(g_n)$, $d(P, g_n)$, and $d(Q, g_n)$ are each smaller than the smaller of $d/2$ and $1/n$. For each n there exists a finite chain $F_n = x_{1n}$, x_{2n}, x_{3n}, $\cdots$, $x_{i(n)n}$ of points belonging to g_n such that (1) all except the last point in this chain belong to $\overline{H}$ but the last does not, (2) the distance of x_{1n} from P is less than $1/n$, and (3) each link has a length less than $1/n$. The limiting set F of $\sum F_n$ ($n = 1, 2, 3, \cdots, \infty$) is closed and connected,‖ contains P, and is a sub-set of T, which is, by definition, a maximal connected sub-set of K. The point set $\sum x_{i(n)-1,n}$ ($n = 1, 2, 3, \cdots, \infty$) is compact and has the same limit points as $\sum x_{i(n),n}$ ($n = 1, 2, 3, \cdots, \infty$). These points are common to K_2, T, and B. But B and T have no points in common, and hence $K_2 \subset \overline{H}$.

Further $K_2 \subset T$. For, if e is greater than zero, we can evidently cover T with a set of spherical regions each of which is compact and has a diameter less than e. In terms of these regions we can then define a point set $\overline{H}_e$ which has properties similar to those of $\overline{H}$, and which contains no point whose

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* *A maximal sub-set, having a property P, of a point set M, is a sub-set N of M which has the property P but is not a proper sub-set of any sub-set of M which has this property.*

† Cf. Gross, p. 810, and Fréchet I, pp. 22 and 26.

‡ That is, B is the set of all limit points of H that do not belong to H.

§ Let h be the complement of $\overline{H}$.

‖ Cf. Janiszewski, loc. cit., p. 98, lines 1–12.

distance from T is greater than e. As above, $K_2 \subset \overline{H}_e$. Then $K_2 \subset T = \prod \overline{H}_{1/i}$ $(i = 1, 2, 3, \cdots, \infty)$.

It follows from the hypothesis of the theorem that K_2, and hence T, contains points of K_1. By an argument analogous to that used in showing that $K_2 \subset T$ we can show that $K_1 \subset T$.

If there exists a point Y of $K - T$, there exists an infinite sub-collection G_Y of G having a limit set containing Y. It follows from the hypothesis of the theorem that G_Y contains an infinite sub-collection having a limit set containing a point of K_1. We have shown above that the limiting set of such a collection is a sub-set of T. Hence the point Y does not exist.

From $T \subset K$ and $K \subset T$ it follows that $K = T$. But this contradicts our supposition that K is not well chained. Thus we have established the truth of our theorem.

THEOREM 12. *In order that a compact point set in a locally compact space D should be the closed and connected limiting set of a collection of well chained point sets, it is necessary and sufficient that it (1) be a maximal well chained sub-set of the limiting set of this collection and (2) contain points in common with the limiting set of every infinite sub-collection of the given collection which has a non-vacuous limiting set.*

It is a consequence of the definition of well-chainedness that if P is a limit point of a point set M then $M + P$ is well chained. Let K be the limiting set of a collection of point sets and K_1 be a maximal well chained sub-set of K. By Theorem 1, K is closed. Since K_1 is a *maximal* well chained sub-set of K, it is closed. The argument used in proving Theorem 11 shows that the condition is sufficient. Its necessity is obvious.

If we leave off the condition that the space D under consideration be locally compact, the condition given in Theorem 12 is not sufficient. Consider, for example, the space D_3 mentioned in the discussion following the statement of Theorem 11. The point A satisfies the condition but is not the limiting set of any sub-collection of $\sum H_{1/i}$ $(i = 1, 2, 3, \cdots, \infty)$. The point just raised applies also to each of the following two theorems.

THEOREM 13. *If $G = [g]$ is a collection of point sets in a separable, locally compact space D, K is the limiting set of G, $f(g) \rightarrow 0$ as $l(g, K) \rightarrow 0$, and K_1 is a maximal well chained sub-set of K, then there exists a countable sub-collection of G whose limiting set is K_1.*

Let P be a point of K_1. There exists a sub-collection G_P of G whose limit set exists and contains P. Suppose that there exists a point Q of K_P, the limiting set of G_P, such that Q is not a point of K_1. By Theorem 11, P and

Q are well chained in K_P. But then K_1 is not a maximal well chained sub-set of K. Thus, the supposition that K_P is not a sub-set of K_1 leads to a contradiction. The truth of the theorem follows by Theorem 9.

THEOREM 14. *If G is a collection of connected point sets in a compact space D and K_1 is a maximal connected sub-set of the limiting set of G, there exists a countable sub-collection of G whose limiting set is K_1.*

DEFINITION. A collection G of point sets is said to be a *separable collection* provided that it contains a countable sub-collection H such that if g is an element of G it is either an element of H or the limiting set of some infinite sub-collection of H.

THEOREM 15. *If G is a collection of closed point sets in a separable space D, a sufficient condition that G be a separable collection is that it contain a sub-collection H such that* (1) *H has the same limiting set as G, and* (2) *is upper semi-continuous with respect to G. In particular, if the collection G just mentioned is upper semi-continuous, it is a separable collection.*

The condition mentioned in the theorem is not necessary. Consider, for instance, the collection G whose elements are the graphs in the euclidean plane of $x = 0$ and of $y = nx$, where $n = 1, 2, 3, \cdots$.

Let K be the common limiting set of G and H. By Theorem 2, G is the sum of a pair of sub-collections G_1 and G_2, and H of a pair of sub-collections G_3 and G_4 such that G_1 and G_3 contain at most a countable number of elements and every element of G_2 and of G_4 respectively is a sub-set of K. Since the space is separable, K is separable and contains a countable sub-set $T = P_1 + P_2 + P_3 + \cdots$ such that every point of it either belongs to T or is a limit point of T. For each positive integer n there exists a countable sub-collection H_n of H having a limit set containing P_n. Let $G_5 = \sum H_n$ ($n = 1, 2, 3, \cdots$, ∞). If g is an element of G_2, it contains a countable sub-set $T_g = Q_{1g} + Q_{2g} + Q_{3g} + \cdots$ such that every point of g either belongs to T_g or is a limit point of it. For every positive integral i the point Q_{ig} is a point or a limit point of a sub-sequence of T; it can easily be shown that Q_{ig} belongs to the *limit* set of a sequence M_{ig} belonging to G_5 such that no element of M_{ig} is at a lower distance from Q_{ig} greater than $1/i$. By the note following Theorem 10, g is the limiting set of $\sum M_{ig}$ ($i = 1, 2, 3, \cdots$, ∞). Hence $G_1 + G_5$ is a sub-collection of G of the type required to make G a separable collection.

THEOREM 16. *The aggregate of all closed point sets in a space L may itself be regarded as a space L provided that "the limit of a sequence of elements" in this hyperspace be defined as the minimal closed set in the given space L containing the non-vacuous sequential limiting set of the given sequence.*

If M is a *closed* point set in a space L, the sequential limiting set of the sequence $M, M, M, M, \cdots$, all of whose elements are the point set M, is clearly M. Thus, the third condition in the definition of a space L is satisfied in our hyperspace. If N is a point set in a space L, there exists[*] a point set K which is the minimal closed point set in that space containing N. From the results just indicated and the definition of a sequential limiting set it follows that our hyperspace satisfies the other requirements of a space L.

THEOREM 17. *The aggregate of all closed point sets in a space S is a space L provided that "the limit of a sequence" is defined as the non-vacuous sequential limiting set of the sequence in the given space S.*

The theorem follows from Theorems 1 and 16.

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[*] Cf. Hausdorff II, p. 230.

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